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A model of unsteady subsonic flow with acoustics excluded

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Diverse subsonic initial-boundary-value problems (flows in a closed volume initiated by blowing or suction through permeable walls, flows with continuously distributed sources, viscous flows with substantial heat fluxes, etc.) are considered, to show that they cannot be solved by using the classical theory of incompressible fluid motion which involves the equation $\operatorname{div} \boldsymbol{u} = 0$. Application of the most general theory of compressible fluid flow may not be best in such cases, because then we encounter difficulties in accurately resolving the complex acoustic phenomena as well as in assigning the proper boundary conditions. With this in mind a new *non-local* mathematical model, where $\operatorname{div} \boldsymbol{u} \neq 0$ in the general case, is proposed for the simulation of unsteady subsonic flows in a bounded domain with continuously distributed sources of mass, momentum and entropy, also taking into account the effects of viscosity and heat conductivity when necessary. The exclusion of sound waves is one of the most important features of this model which represents a fundamental extension of the conventional model of incompressible fluid flow. The model has been built up by modifying both the general system of equations for the motion of compressible fluid (viscous or inviscid as required) and the appropriate set of boundary conditions. Some particular cases of this model are discussed. A series of exact time-dependent solutions, one- and two-dimensional, is presented to illustrate the model.

1. Introduction

The main theoretical basis of incompressible fluid mechanics looks so complete today that any attempt to change or expand the well-known fundamental concepts could be regarded as unpromising. Nevertheless, let us discuss some features of the classical model of incompressible fluid flow that has been comprehensively expounded along with numerous applications by Lamb (1932), Landau & Lifshitz (1959), Milne-Thomson (1960), Batchelor (1967), Panton (1984).

According to the usual interpretation of the inviscid version of that model, the general continuity equation in the absence of mass sources

$$\partial\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0 \tag{1}$$

is split into two equations:

$$\partial\rho/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\rho = 0, \tag{2}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0. \tag{3}$$

After adding the equation of motion for inviscid fluid with external mass force $\boldsymbol{f}$

$$\rho(\partial\boldsymbol{u}/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u}) + \boldsymbol{\nabla}p - \rho\boldsymbol{f} = 0, \tag{4}$$

we obtain a closed system (2)–(4) for the local variations of fluid velocity $\boldsymbol{u}$, density ρ and static pressure p which is here determined to within an arbitrary additive function $p_a(t)$. This system has remained unchanged since the eighteenth century when it was proposed by Euler (1769).

In fact equation (2) implies invariable density for each flowing liquid particle of small fixed volume while the whole density field may have considerable gradients. The density of the liquid particle is thereby supposed to be independent of any pressure variations. This key assumption, although seeming too strong, represents the rigorous definition of an incompressible fluid medium, and, as a result, enables us to derive the closed system of governing equations without using any equation of state (thus it is not necessary to regard system (2)–(4) only as a reduction of the general model of compressible fluid flow when the characteristic Mach number tends to zero). Some authors, without going into detail, identify the term *incompressible fluid flow* with a constant density all over the flow region.

In this way the classical model of incompressible fluid flow includes equation (3) which simplifies any possible subsequent equations and allows us to use all the mathematical methods developed for the analysis of solenoidal vector fields (introducing, in part, the scalar stream function for two-dimensional flow). This explains why one may feel anxiety in considering the idea of giving up equation (3) as a step towards a generalization of the model.

It is now universally recognized that this model, derived from the analysis of fluid motion in a small volume surrounded by an infinitely extended fluid medium, can be applied to the study of any *bounded* flows without any changes in equations (2)–(4). The form of system (2)–(4) is regarded as absolutely independent of the boundary conditions, in spite of the fact that we consider so exotic a medium as an incompressible fluid characterized by an infinitely large speed of sound. However, a series of simple illustrations will be suggested below to show that the usual local model of incompressible fluid flow does not work when applied to the solution of some initial-boundary-value problems.

Let us consider a closed container filled with a gaseous medium and bounded by a permeable non-moving wall, through which subsonic blowing takes place. Evidently, this problem cannot be simulated within system (2)–(4). When a moving impermeable wall, as a boundary, changes the volume of a closed container, the simulation of this situation is also forbidden by equation (3). It is generally impossible to assign values of the normal velocity at each boundary point in an independent manner, if we use equation (3). Only those profiles of the normal velocity are acceptable which have zero integral all over the boundary (if there is no mass source in the volume). Unfortunately, we have to apply too restricted a choice of boundary conditions because of an oversimplified system of differential equations governing the fluid motion in a small *internal* volume. As a consequence, it becomes clear that the classical model should be extended to avoid a number of serious limitations.

If the pronounced effects of viscosity and molecular heat conductivity are under study, one could try to take the usual incompressible version of the Navier–Stokes equations, where $\operatorname{div} \boldsymbol{u} = 0$ is assumed too, supplemented by the equation of heat conduction. Then we can see that additional difficulties arise in applying this ordinary approach. For instance, let us take the case of a container filled with a gaseous medium and bounded by heat-conducting impermeable rigid walls. Under physically possible conditions, on increasing the wall temperature by external heating we can stimulate the convective heat transfer inside as well as the processes of molecular heat conduction. As a result the pressure $P(t)$ averaged over the volume will increase. If we expand the model by introducing the functional link between density and temperature that is so common in various problems of heat convection (including the simplified versions of this approach like the Boussinesq 1903 approximation), the above changes in $P(t)$ cannot be predicted and so a net decrease of total mass in the container will occur. Hence, such a simple version of the thermodynamic equation of state as a direct relation between density and temperature is insufficient for the study of processes accompanied by appreciable time variations of the average pressure in a bounded flow region.

Moreover, we find equation (3) to not apply when there is a substantial heat transfer due to molecular thermal conductivity. Indeed, on using (3) even in a one-dimensional case (one could take a cylinder which is bounded by two rigid impermeable walls with different values of the temperature assigned), we meet with an obvious contradiction: mass transfer, caused only by heat conduction, can take place without any flow! It is also relevant to recall the 'energy-equation paradox' in a heat-conducting incompressible flow, described by Panton (1984, 10.3). There is the vanishing difference between specific heats c_p and c_v at constant pressure and at constant volume respectively (or $\gamma = c_p/c_v \equiv 1$) that in turn prohibits the use of the common set of thermodynamic relations in the study of subsonic gas flows. Thus we come to the undeniable conclusion that any known version of the incompressible viscous flow model is unlikely to be acceptable for the simulation of thermal conduction with significant heat fluxes.

Another topical problem to be solved is associated with the question of whether sources of mass, momentum and entropy, continuously distributed in the flow region, ought to be simulated solely within the general model of *compressible* fluid flow. In practice incompressible fluid mechanics operates only with the notion of a point mass source, when equation (3) is valid everywhere beyond that point. The attendant singularities are the high price we pay for using equation (3) in such cases. Nobody has proposed how to expand the model (2)–(4) taking into consideration the continuously distributed mass sources, because in this case the splitting of the continuity equation is not a trivial procedure. Additional questions appear if thermal phenomena with distributed heat sources take place as well. One should take into account that the distributed mass sources are usually accompanied by sources of entropy since any new fluid particle has a definite specific entropy. Hence, any attempt to introduce distributed mass sources into equation (1) demands consideration of the entropy balance. Sometimes entropy sources may arise without mass production, particularly in the case of volume heat release. In any event, we need a new model which would give the possibility of studying distributed sources of both mass and entropy that could be specified independently.

Here we should also touch upon a long-standing problem of far-reaching importance. In reality any time-dependent sources have to act on a certain unsteady mean flow (or background flow) as well as to generate acoustic disturbances; see Goldstein (1976). It is highly desirable to find conditions where the separation of these different effects is possible within a definite theoretical approach, in order to determine what portion of the source action causes the evolution of the mean flow. Otherwise, one can find sound sources where they are completely absent. Again, similar questions arise if we study the action of quite general unsteady boundary conditions, for instance those assigned on permeable or moving walls.

Obviously the classical model of incompressible fluid flow is inappropriate for the solution of the above problems. One can argue that in all the cases mentioned we should apply the more general (and surely more complicated) model of compressible fluid flow, viscous or inviscid as required, instead of discussing defects of system (2)–(4). But in doing this we meet with inevitable numerous difficulties in the analysis of complex nonlinear acoustic phenomena. Computational methods of simulation can play a significant role here, along with experimental research. However, any mistake in the proper assigning of acoustic impedance of the boundary, a permeable one in particular, can lead to unpredictable consequences in the flow evolution. Sometimes the impossibility of formulating a plausible set of boundary conditions represents an unavoidable obstacle to the solution. To this must be added that we have to use the most sophisticated finite-difference methods of integrating the governing equations to ensure the necessary temporal and spatial resolution of acoustic processes characterized by relatively small amplitudes of sound disturbances. If acoustics is not the main research goal, it is not helpful to simulate simultaneously both the evolution of unsteady mean flow and the attendant acoustic processes since these phenomena, when considered separately, usually have very different characteristic times. During the computational study of subsonic flows this would require an extremely small time step in the finite-difference scheme, which in turn demands a lot of computer time. All these difficulties explain why such general solutions are very rarely met in publications. Nevertheless, we cannot avoid this thorny path when considerable interaction between unsteady subsonic flow and sound waves is under investigation – for instance, if we study self-excited resonance phenomena in ducts of complex geometry, where instability of separated viscous flows occurs along with accompanying processes of sound generation and acoustic feedback (see Fedorchenko 1987).

Therefore it seems attractive to create a radically new mathematical model for the simulation of unsteady subsonic flows in a bounded domain taking into account continuously distributed sources of different kinds, rather complex boundary conditions, substantial heat conduction, etc., but excluding all acoustic effects. It would imply that we are looking for a model which is much more general than (2)–(4) but less general than the model of compressible fluid flow. Thus we could fill a sizable gap in the available mathematical models of fluid mechanics. For the flow in a bounded spatial domain this approach would mean that the medium is compressible as a whole, with considerable time changes of average pressure being allowed, but the existence of sound waves is completely precluded. In doing so we assume the sound speed to be infinitely large even if we retain something like a thermodynamic equation of state. Mathematically this means that we need a time-dependent system of governing equations that would display the local properties as partially elliptic (as well as, additionally, parabolic ones in the case of unsteady viscous flow) but it would have no hyperbolic characteristics responsible for the propagation of sound (only convection by the flow is permitted). It should be emphasized that we do not demand this system to be of strictly differential type, local in both time and space, since we are trying to take into account some non-local effects resulting from boundary conditions.

It is well known that the filtering of sound disturbances, especially high-frequency waves, represents a problem of much current interest in the computational simulation of numerous flows, both internal and free, when acoustics is not the research focus. Moreover, boundary conditions assigned in an incorrect manner (for instance, boundary value overspecification) may be regarded as a source of such waves which in turn, emanating from the boundary, cover all the domain with error (see Oliger & Sundstrom 1978). A great variety of approaches has been proposed to settle this problem: the introduction of additional dissipative terms into the basic equations, special finite-difference approximations, sound absorption on the boundary, etc. But all these methods are far from universal and seldom work in a fully satisfactory manner since it is hardly possible to exclude all sound sources when we use the general model of a compressible medium. It seems that the modification of subsonic flow equations to eliminate completely the mechanism of sound propagation could be the most radical and effective approach for removing all above difficulties caused by acoustic effects. The resulting system of course must remain quite general in order to describe any other phenemona under research. However, we do not simply pursue the aim of getting rid of acoustics everywhere if possible. On the contrary, it is clear that only by separating out all 'incompressible features' can one come to a better understanding of sound generation and propagation in unsteady subsonic flows.

The results given below represent a fundamental extension of the classical model of incompressible fluid flow. This new non-local model, where $\operatorname{div} \boldsymbol{u} \neq 0$ in general, has been developed for the simulation, either theoretical or computational, of unsteady subsonic flows in a bounded spatial domain with the presence of continuously distributed sources of mass, momentum and entropy as well as with substantial heat fluxes, but excluding acoustics. It is shown that this model does broaden significantly the class of initial-boundary-value problems in fluid mechanics that could be solved without considering acoustic phenomena. The central ideas of this approach were briefly described by the author in 1995 and 1996.

The paper is divided into three main sections as follows. In the next we describe a logical way of deriving the new approximate system of governing equations by modifying the general equations commonly used for the study of inviscid compressible flows within a certain initial-boundary-value problem. The set of basic assumptions as well as the main properties of the derived equations are discussed. The important particular case of adiabatic flow is investigated.

In the third section the additional effects of viscosity and heat conductivity are considered. It is shown how these effects change the system of basic equations formerly derived for the inviscid case. Some new features are discussed within this extended version of the model.

In the fourth section a set of time-dependent two-dimensional exact solutions is found for near-adiabatic subsonic flows of both viscous and inviscid media. The family of exact one-dimensional solutions has also been found for subsonic flows of heat-conducting viscous gas without distributed sources.

2. Inviscid subsonic flow

First we shall consider the general system of local equations which could be applied to the simulation of unsteady subsonic flow of a compressible ideal medium in a certain bounded spatial domain G within a finite time interval $J = (0, t_f)$:

$$\partial \rho \boldsymbol{u}/\partial t + (\rho \boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \boldsymbol{u} \nabla \cdot (\rho \boldsymbol{u}) + \nabla p_c = \rho \boldsymbol{f} + \boldsymbol{w}, \tag{5}$$

$$\partial \rho/\partial t + \nabla \cdot (\rho \boldsymbol{u}) = \xi \rho, \tag{6}$$

$$\partial s/\partial t + \boldsymbol{u} \cdot \nabla s = q, \tag{7}$$

$$F(s, p_c, \rho) = 0. \tag{8}$$

Here we denote the time, velocity of fluid particles, density, static pressure, specific entropy, assigned mass force, rate of momentum change because of mass source, and mass source strength per unit mass as t, $\boldsymbol{u}$, ρ, p_c, s, $\boldsymbol{f}$, $\boldsymbol{w}$, ξ respectively (in particular $\boldsymbol{w} = \xi \rho \boldsymbol{u}$ if fluid particles, arising from mass sources, are at rest relative to the local background flow), q is the entropy source per unit mass due to both volume heat release and a non-zero mass source (the case is possible when $\xi \neq 0$ but $q = 0$ if the arising fluid particles have the local specific entropy of the mean flow). Naturally equation (7) is equivalent to the following one:

$$\partial(\rho s)/\partial t + \nabla\cdot(\rho s \boldsymbol{u}) = Q_s = \xi\rho s + \rho q.$$

To begin with, we assume that ξ, q, $\boldsymbol{f}$, $\boldsymbol{w}$ are independently assigned functions of $\boldsymbol{r}$, t in $G\times J$, where $\boldsymbol{r}$ is the radius-vector of a point considered in a fixed coordinate system. All the above-mentioned variables are supposed to be continuous and smooth functions of $\boldsymbol{r}$, t in $G\times J$. The medium will, to fix the ideas, be regarded as a perfect gas, i.e. we take (8) as

$$s = c_v \ln(p_c/\rho^\gamma), \quad \gamma = c_p/c_v = \text{const},$$

where c_p, c_v are the values of specific heats at constant pressure and constant volume respectively.

While posing an evolution problem in $G\times J$ one should specify a set of initial and boundary conditions. We shall take the following local relations as possible boundary conditions:

$$\Phi_j(u_n, p_c, s, \boldsymbol{r}, t) = 0, \quad \boldsymbol{r}\in\Gamma, \quad t\in J, \quad j = 1, \ldots, N, \tag{9}$$

where $u_n = \boldsymbol{u}\cdot\boldsymbol{n}$, $\boldsymbol{n}$ is the outward normal to the smooth boundary surface Γ, which may move, and Φ_j are assigned functions. Of course, the concrete form of Φ_j as well as the total number N of independent boundary relations at the given boundary point depend on the sign of $u_f = u_n - u_\Gamma$ where u_Γ denotes the specified velocity of $\Gamma(t)$ along $\boldsymbol{n}$. For instance, the set of boundary conditions

$$\Phi(u_n, p_c, \boldsymbol{r}, t) = 0, \tag{10}$$

$$s = \theta(\boldsymbol{r}, t), \tag{11}$$

can be applied at $\boldsymbol{r}\in\Gamma$, where $u_f < 0$. Relation (11) means that the entropy of inflowing fluid particles should be prescribed at each boundary point where $u_f < 0$. On the permeable part of the boundary where $u_f > 0$ we should use only one condition, like (10).

Below we shall emphasize, as important particular cases, two kinds of domain G in accordance with a possible application of two types of boundary conditions:

$G = G_m$, if the normal velocity u_n is assigned all over Γ (then $\partial\Phi/\partial p_c = 0$ in (10));

$G = G_p$, if the pressure p_c is assigned on the permeable surface $\Gamma_p \subset \Gamma$ ($p_c = p_\infty =$ const in the limiting case of an infinitely remote boundary Γ_p).

In more general cases this separation can correspond to the use of two models of permeable boundaries S_m and S_p proposed by the author in 1982 and 1986.

Attention should be drawn to the fact that all these boundary conditions in no way influence the form of equations (5)–(8), since we are considering the most general model yet of compressible fluid flow. However, we shall further depart from this concept in our efforts to build the new non-local model of subsonic flow without acoustic effects.

As a first step, we split the pressure in the manner

$$p_c(\boldsymbol{r}, t) = P(t) + p(\boldsymbol{r}, t), \tag{12}$$

where $P(t)$ is a certain average pressure in G that, as shown below, depends on ξ, q as well as on the assigned set of initial and boundary conditions. For p a normalization condition is introduced,

$$\int_G p\,\mathrm{d}v = 0 \quad \text{or} \quad p(\boldsymbol{r}_o, t) = 0, \tag{13}$$

where $\boldsymbol{r}_o$ denotes a particular fixed point of Γ_p. It should be noted that this does not correspond to the usual decomposition of a flow variable as the sum of a known mean value and a small disturbance to be found, but here both P and p are unknown variables in general.

Some estimates are assumed to be valid:

$$\left.\begin{aligned} P &= O(\rho_o a^2), \quad |p| \leqslant \delta p, \\ p &\approx p_1 + p_2, \quad \|p/P\|_G = \delta p/P = \epsilon \approx \epsilon_1 + \epsilon_2, \\ \epsilon_1 &= \|p_1/P\|_G \lesssim M^2 = U^2/a^2 \ll 1, \quad U = \|\boldsymbol{u}\|_G, \\ \epsilon_2 &= \|p_2/P\|_G = M_a \ll 1, \\ \delta p(t) &= \delta p_c(t) = \max|p_c(\boldsymbol{r}_1, t) - p_c(\boldsymbol{r}_2, t)|, \quad \boldsymbol{r}_1, \boldsymbol{r}_2 \in G, \quad t \in J; \end{aligned}\right\} \tag{14}$$

ρ_o, U, a are respectively the characteristic values of density, flow velocity and adiabatic speed of sound in $G \times J$; by introducing p_1, p_2 we are trying to separate approximately the contributions of local dynamic effects, often essentially nonlinear, in unsteady 'background flow' (in part p_1 reflects the degree of flow non-uniformity since $p_1 = 0$ in uniform flow at any values of the mean Mach number M) from acoustic disturbances of pressure which are assumed to be manifested by small values of acoustic Mach number M_a. Sometimes it is better to specify the characteristic velocity as follows: $U^2 = LW$, where $W = \|\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u}\|_G$ and L is the characteristic length. Evidently, the source terms in (6)–(7) do not change these estimates when

$$|\xi| \lesssim O(U/L), \quad |q/s| \lesssim O(U/L).$$

For instance, if we consider the gravity force $\boldsymbol{f}_g$, the hydrostatic effects can contribute to δp_c and hence the requirement

$$(\delta p_c/P)_g = O(\rho_o g h/P) < O(\epsilon) \ll 1,$$

should be met, where h denotes the size of domain G along vector $\boldsymbol{f}_g$, and $g = |\boldsymbol{f}_g|$ is the acceleration due to gravity. Thus we arrive at some limitations on h to retain the validity of our model in this case. However, as will be mentioned below, internal subsonic flows with reasonable h represent the main application area of this model. Moreover, we can take G with $L \gg h$ where L is the maximum length of G.

It is logical to assume $|\mathrm{d}P/\mathrm{d}t| \lesssim O(\rho_o a^2/\tau)$ where $\tau = O(L/U)$. Then

$$\begin{aligned} Sh_u &= \|\mathrm{d}P/\mathrm{d}t\|\,\tau/P \lesssim O(1), \\ Sh_a &= \|\mathrm{d}P/\mathrm{d}t\|\,\tau_a/P \lesssim O(M) \ll 1. \end{aligned}$$

A small value of Sh_a implies that the characteristic time for appreciable changes in $P(t)$ is much longer than the time interval $\tau_a = O(L/a)$ during which a sound wave crosses the flow region in real conditions.

Taking into account (12)–(14) we shall use the following form of the equation of state instead of (8):

$$F(s(\boldsymbol{r}, t), P(t), \rho(\boldsymbol{r}, t)) = 0, \tag{15}$$

which approximates (8) with an error $O(\epsilon)$. It means that the variations of both density and entropy, which can be quite substantial, are not influenced by the relatively small spatial changes of pressure. In doing so, we have to be aware that this may introduce errors $O(\epsilon)$ into the values of all flow variables. Nevertheless, this assumption is more general than that used as basic in the classical model of incompressible fluid flow. Actually, in the latter the density is supposed to be independent of any pressure variations.

After using (15) and substituting (12) into (5) the following system of equations is suggested for the simulation of unsteady subsonic flow in $G \times J$:

$$\partial \rho \boldsymbol{u}/\partial t + (\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) + \boldsymbol{\nabla} p = \rho \boldsymbol{f} + \boldsymbol{w}, \tag{16}$$

$$\partial \rho/\partial t + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = \xi \rho, \tag{17}$$

$$\partial s/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla} s = q, \tag{18}$$

$$F(s, P, \rho) = 0. \tag{19}$$

Note that the form of (16)–(18) is exactly the same as that of (5)–(7), but the appearance of the modified equation (19) results in a model with new unusual features. In particular, it will be shown below that this new model does filter all sound waves.

This sytem must be supplemented by boundary conditions (9), where $p_c = P + p$, although in many cases it is quite possible to omit p in (9), giving

$$\Phi_j(u_n, P, s, \boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in \Gamma, \quad t \in J, \quad j = 1, \ldots, N, \tag{20}$$

and so boundary relations (10), (11) become

$$\Phi(u_n, P, \boldsymbol{r}, t) = 0, \quad s = \theta(\boldsymbol{r}, t), \quad \boldsymbol{r} \in \Gamma, \quad t \in J.$$

We shall use these simplified conditions because of the obvious advantage that now the variable p is absent everywhere except in (16), if functions $\boldsymbol{f}$, $\boldsymbol{w}$, ξ, q are independent of p. Then only the values of $\boldsymbol{\nabla} p$ are significant.

To make system (16)–(19) closed, we should also specify the procedure for finding the *global flow variable* $P(t)$ which is generally unknown. With this aim we take the differential form of the equation of state (19)

$$\partial s/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla} s = (\partial s/\partial P)\,\mathrm{d}P/\mathrm{d}t + (\partial s/\partial \rho)\,[\partial \rho/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla} \rho]. \tag{21}$$

In accordance with (19) we have now $s = c_v \ln (P/\rho^{\gamma})$, $\partial s/\partial P = c_v/P$, $\partial s/\partial \rho = -c_p/\rho$. By using (17), (18) along with (21) we find that

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \xi + \eta - (\mathrm{d}P/\mathrm{d}t)/(\gamma P), \quad \eta = q/c_p. \tag{22}$$

Comparing this equation with more general version derived exactly from (6)–(8)

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \xi + \eta - (\mathrm{d}P/\mathrm{d}t + \partial p/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla} p)/(\gamma (P + p)),$$

one can see that here not only has the small value of p been omitted in the expression $P + p$, but we have also simplified the general form by assuming

$$|(\partial p/\partial t + \boldsymbol{u} \cdot \boldsymbol{\nabla} p)/P| = O(\epsilon U/L) \ll |\boldsymbol{\nabla} \cdot \boldsymbol{u}| = O(U/L).$$

If we integrate (22) over the confined domain G, which has a moving boundary $\Gamma(t)$ and accordingly a changing volume $V(t)$, the following equation can be derived with the use of Gauss' theorem:

$$\frac{V}{\gamma P}\frac{\mathrm{d}P}{\mathrm{d}t} = \int_G (\xi + \eta)\,\mathrm{d}v - \int_\Gamma u_n\,\mathrm{d}\sigma. \tag{23}$$

Rewriting the new form of (10) in accordance with (20) as

$$u_n = \quad (P, \boldsymbol{r}, t), \quad \boldsymbol{r} \in \Gamma, \quad t \in J, \tag{24}$$

and substituting (24) into (23) we obtain the integro-differential equation allowing us to determine $P(t)$ in the general case.

Equations (23), (24) explain why we cannot neglect the term $\mathrm{d}P/\mathrm{d}t$ in (22) even if the non-dimensional value $|\mathrm{d}P/\mathrm{d}t|\,L/(PU)$ is quite small. Indeed, the variations of p_1, p_2

may produce a very small contribution into the total mass balance in G, much less after averaging over the volume and within time interval $\tau \gg \tau_a$. On the contrary, even the slowly varying pressure $P(t)$, for instance, is monotonically increasing due to non-zero mass flux through the boundary, it can cause drastic changes in the total mass value. Moreover, this simple example emphasizes the fact that just due to the term $\mathrm{d}P/\mathrm{d}t$ in (22) we can assign the boundary values of normal velocity in a quite independent manner, and in turn this equation can be basic for the calculation of $P(t)$.

When we investigate the particular case of domain $G_m(\partial \quad /\partial P = 0)$, it is easy to give an explicit formula for $P(t)$. Denoting the right-hand side of (23) as $Z(t)$, we have

$$P(t) = P_o \exp \int_0^t \gamma(Z/V)\,\mathrm{d}t, \quad P_o = P(0). \tag{25}$$

If we consider the special case $G = G_p$, then we take $P(t) = p_c(\boldsymbol{r}_o, t)$, $\boldsymbol{r}_o \in \Gamma$ as an assigned function and so we do not need equation (23) (although now one could evaluate the total fluid flux through Γ_p by means of (23), at least in the adiabatic case). Here we find the average pressure $P(t)$ in G_p by assigning the mean pressure $P_e(t)$ on Γ_p (more accurately $P(t) = P_e(t) + O(\epsilon)$, $P_e(t) = p_c(\boldsymbol{r}_o, t) + O(\epsilon)$). It means that we do not prescribe the average pressure in G regardless of the boundary conditions, as is often done.

It should be noticed, however, that the simplified boundary conditions (20) may not be accepted in some cases, especially if we consider the topical problem of formulating the set of appropriate conditions on the outflow boundary Γ_e where the mean pressure $P_e(t)$ is assigned and through which a subsonic flow carrying strong vortices escapes from domain G. For instance, such a boundary could simulate the exit section of a confined duct that is connected to a large volume filled with gaseous medium under the given mean pressure $P_e(t)$. Some methods were given by the author in 1986 for solving this problem for the computational simulation of both viscous and inviscid compressible flows. These provided an acceptable resolution of self-generated acoustic phenomena, although the non-local procedures for active boundary control proposed there may cause extreme difficulties in the computational realization of the initial-boundary-value problem. Principally the same approach can be used here by applying the generalized non-local version of (9).

Some additional complexity can arise if ξ, η are functions of $\rho, s, \boldsymbol{r}, t$. All the above generalizations result in a rather complex system of integro-differential type. Nevertheless, the solution of a definite initial-boundary-value problem is quite possible, at least with the use of computational methods.

In conclusion, we have built a closed non-local system of equations for the full set of flow variables $\{\boldsymbol{u}, P, p, \rho, s\}$.

We can also obtain the equivalent system in which entropy is excluded:

$$\partial \rho \boldsymbol{u}/\partial t + (\rho \boldsymbol{u} \cdot \boldsymbol{\nabla})\,\boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla}(\rho \boldsymbol{u}) + \boldsymbol{\nabla} p = \rho \boldsymbol{f} + \boldsymbol{w}, \tag{26}$$

$$\partial \rho/\partial t + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = \xi \rho, \tag{27}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = \xi + \eta - (\mathrm{d}P/\mathrm{d}t)/(\gamma P), \tag{28}$$

supplemented by the above procedure for the determination of $P(t)$, although while solving (26)–(28) one should keep in mind the associated changes in boundary conditions (11) as well as the physical meaning of the source η. On obtaining both P and ρ, it is easy to find s from (19) and in turn to evaluate the temperature $T = P/\rho R$ ($R = c_p - c_v = \mathrm{const}$) or any other thermodynamic function.

We could even exclude P from system (26)–(28) if the domain G_m is under consideration. When we use (23), equation (28) can be rewritten in the following integro-differential form:

$$\nabla \cdot \boldsymbol{u} = \xi + \eta - Z/V.$$

This compact equation reflects the most remarkable feature which distinguishes this model from any traditional approaches: the system of equations governing the fluid motion in any small internal volume is instantaneously non-locally related to the boundary conditions, as well as to the distribution of sources all over the flow domain. This is not the usual instantaneous influence of boundary conditions on the local fluid motion that is inherent in the classical incompressible flow model due to its partially elliptic nature, but it is an explicit non-local dependence of the basic equations on both boundary conditions and distributions of volume sources.

One can see that only in the particular case when $\mathrm{d}P/\mathrm{d}t = \xi = \eta = 0$, does our model reduce to the classical model of incompressible fluid flow where $\nabla \cdot \boldsymbol{u} = 0$, $s = s(\rho)$.

It is very important that this non-local model enables us to simulate unsteady subsonic flows with continuously distributed sources of mass, momentum and entropy as well as to assign the boundary values of normal velocity in an independent manner. In contrast, the usual local model of incompressible fluid flow cannot describe such general cases.

We can perform the local characteristic analysis of system (26)–(28) following the approach developed by Courant & Hilbert (1962). Let us take $\boldsymbol{w} = \xi\rho\boldsymbol{u}$ and assume $\boldsymbol{f}$, ξ, η, P to be known functions within a small internal volume. Then we write the quasi-linear version of (26)–(28) for independent flow variables $\{\boldsymbol{u}, \rho, p\}$ as

$$\partial u_i/\partial t + u_k \partial u_i/\partial x_k + (\partial p/\partial x_i)/\rho = f_i, \tag{29}$$

$$\partial \rho/\partial t + u_k \partial \rho/\partial x_k = m, \tag{30}$$

$$\partial u_k/\partial x_k = g, \quad i, k = 1, 2, 3, \tag{31}$$

where f_i, $m = \rho(\xi - g)$, $g = \xi + \eta - (\mathrm{d}P/\mathrm{d}t)/(\gamma P)$ represent zero-order forcing terms which do not change the main local properties of (29)–(31). To classify this system we should investigate the matrix partial differential operator corresponding to (29)–(31). On doing so we find the characteristic algebraic equation (characteristic cone) for new variables $\{\tau, \theta_i\}$ instead of $\{t, x_i\}$

$$(\tau + u_k \theta_k)^3 (\theta_k^2 + \tau^2) = 0.$$

This equation shows that system (29)–(31) displays combined local properties: partially hyperbolic (three sets of characteristics $\mathrm{d}x_i/\mathrm{d}t = u_i$ responsible for the convection by flow) and partially elliptic (instantaneous global connection between fields of pressure and velocity). These resemble the features of the traditional system of equations applied to the simulation of incompressible fluid flows. Our model permits us to study the evolution of two types of disturbances, vorticity and entropy (in problems of hydrodynamic stability too), but excluding sound waves. The absence of sound waves is obvious just from the modified equation of state (19) where p is omitted, i.e. we have no explicit connection between local variations of p and ρ; this results in an infinitely large speed of sound. However, the significant aspect of this model is that it is possible to estimate the values of $a^2 = \gamma P/\rho$ and hence to calculate the characteristic value of $M_o = U/a_o$. Actually the value of M_o is assigned by the initial value of average pressure P_o. In turn one can evaluate immediately the allowable accuracy of a solution. Thus we can regard the pre-assigned dimensionless parameter (Euler number) $Eu_o = P_o/(\rho_o U^2) = 1/(\gamma M_o^2) \gg 1$ as a principal criterion of similarity within this model.

Let us now consider the important particular case of this model when $\xi = 0$, $\eta = 0$, $s = \text{const}$. Then we write the following system instead of (26)–(28):

$$\partial \boldsymbol{u}/\partial t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + (\boldsymbol{\nabla} p)/\rho = \boldsymbol{f}, \tag{32}$$

$$\mathrm{d}\rho/\mathrm{d}t + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = 0, \tag{33}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = -(\mathrm{d}P/\mathrm{d}t)/(\gamma P) = \phi(t), \tag{34}$$

$$\rho = \rho(t) = \rho_o(P/P_o)^{1/\gamma}, \quad T = T(t) = T_o(P/P_o)^{(\gamma-1)/\gamma}, \tag{35}$$

where $P_o = P(0)$, $T_o = T(0)$, $\rho_o = \rho(0)$. Here the variables $P(t)$, $\phi(t)$ depend only on the boundary conditions according to (23).

By introducing $\boldsymbol{\Omega} = \operatorname{rot}\boldsymbol{u}$ we can derive from (32)

$$\partial\boldsymbol{\Omega}/\partial t + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\,\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} + \phi\boldsymbol{\Omega} = \operatorname{rot}\boldsymbol{f}. \tag{36}$$

In the two-dimensional case we have

$$(\boldsymbol{\Omega}\cdot\boldsymbol{\nabla})\,\boldsymbol{u} = 0, \quad \boldsymbol{\Omega} = \{0, 0, \zeta\}, \quad \zeta = \partial v/\partial x - \partial u/\partial y,$$
$$\operatorname{rot}\boldsymbol{f} = \{0, 0, \omega\}, \quad \omega = \partial f_y/\partial x - \partial f_x/\partial y$$

and then equation (36) reduces to

$$\partial\zeta/\partial t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\zeta + \phi\zeta = \omega \tag{37}$$

or, in another manner,

$$\partial\zeta/\partial t + \boldsymbol{\nabla}\cdot(\zeta\boldsymbol{u}) = \omega. \tag{38}$$

Equations (36), (37) differ significantly from the routine vorticity equations used in the classical model of incompressible fluid flow. The term $\phi\zeta$ in (37) can be regarded as a source of vorticity. Due to the global parameter $\phi(t)$ the linear local dependence of this term on the vorticity should be emphasized, and so it may play a decisive role in problems where the instability of shear flows is under study. Consequently, a new method of boundary control over unsteady subsonic flow can be offered: it is possible to stabilize the flow by changing the boundary conditions, and hence $P(t)$, in the appropriate manner. Apparently, by decreasing P ($\mathrm{d}P/\mathrm{d}t < 0$, $\phi > 0$) with the other conditions the same, we could promote the stability of shear flow.

Now we shall pose an initial-boundary value problem for isentropic flow ($\boldsymbol{f} = 0$) in the confined plane duct $G = \{0 < x < L, 0 < y < H\}$ with impermeable sidewalls ($v(x, 0, t) = v(x, H, t) = 0$). Two kinds of boundary conditions can be assigned at the ends:

(*a*) $u(0, y, t) = U_1(y, t)$, $u(L, y, t) = U_2(y, t)$,
(*b*) $u(0, y, t) = U_b(y, t)$, $P = P_{x=L}(t)$ (i.e. here we can demand $p(L, 0, t) = 0$).

The initial conditions could be taken as follows

$$\boldsymbol{u}(x, y, 0) = 0, \quad P(0) = P_o, \quad x, y \in G$$

and then we have to assume $U_1(y, 0) = U_2(y, 0) = U_b(y, 0) = 0$. Applying (23), (25) we find for case (*a*) that

$$\phi(t) = (m_2 - m_1)/(HL), \quad m_k(t) = \int_0^H U_k(y, t)\,\mathrm{d}y, \quad k = 1, 2,$$

$$P = P_o \exp\int_0^t (-\gamma\phi)\,\mathrm{d}t.$$

In case (*b*) both functions $P(t)$ and $\phi(t)$ are known.

Now we can decompose the velocity field in G as

$$\boldsymbol{u} = \boldsymbol{u}_p + \boldsymbol{u}_s, \quad \operatorname{rot} \boldsymbol{u}_p = 0, \quad \operatorname{div} \boldsymbol{u}_s = 0,$$
$$u_p = \phi x, \quad v_p = 0, \quad \zeta = \zeta_s,$$

and so it is possible to introduce the stream function $\psi(x, y, t)$ such that

$$u_s = -\partial\psi/\partial y, \quad v_s = \partial\psi/\partial x, \quad \Delta\psi = \zeta. \tag{39}$$

From (37) we derive

$$\partial\zeta/\partial t + u_s \partial\zeta/\partial x + v_s \partial\zeta/\partial y + \partial[x\phi\zeta]/\partial x = 0. \tag{40}$$

Thus we have obtained the closed system of equations that governs the flow in G.

As a simple example we can find the exact solution of the unsteady problem where

$$u = Ug + \phi x, \quad v = 0, \quad U = U(y), \quad g = g(t). \tag{41}$$

Substituting (41) into (40) we arrive at the conclusion that functions g and ϕ must be connected by the equation

$$\mathrm{d}g/\mathrm{d}t + \phi g = 0$$

that is equivalent to the relation

$$g = g_o[P/P_o]^{1/\gamma}, \quad g_o = g(0), \tag{42}$$

which is valid for any profile $U(y)$. For instance, it could correspond to case (b) when we assign $P(t)$ as well as $u(0, y, t) = Ug$ in accordance with (42).

We can also consider one-dimensional adiabatic subsonic flow in a cylinder under the boundary conditions of either type (a) or (b) at the butt-ends. With this aim we write the one-dimensional version of system (32)–(34) for flow variables $\{u, p, P, \rho\}$:

$$\rho[\partial u/\partial t + u\,\partial u/\partial x] + \partial p/\partial x = 0, \tag{43}$$
$$\mathrm{d}\rho/\mathrm{d}t + \rho\,\partial u/\partial x = 0, \tag{44}$$
$$\partial u/\partial x = -(\mathrm{d}P/\mathrm{d}t)/(\gamma P). \tag{45}$$

Here we can keep a rather simple method of solution: first $u(x, t)$, $\rho(t)$, $P(t)$ can be found by using equations (44), (45) as well as the assigned set of initial and boundary conditions; then we shall get $p(x, t)$ from (43), (13). Such a method is however valid only in the one-dimensional case.

In case (a) where $U_1 = U_1(t)$, $U_2 = U_2(t)$, it is easy to find the explicit form

$$u = U_1 + x(U_2 - U_1)/L,$$
$$P = P_o \exp \int_0^t [\gamma(U_1 - U_2)/L]\,\mathrm{d}t, \quad \rho = \rho_o(P/P_o)^{1/\gamma}.$$

The linear distribution of velocity along the cylinder between arbitrary boundary values U_1 and U_2 represents the main feature of this flow. Of course this flow evolution differs from the case of the fully compressible flow model where the variable boundary values of the normal velocity set up unsteady mean flow, but generate sound waves as well.

Then we determine $p(x, t)$ by integrating (43); this can be done by using the normalization condition

$$\int_0^L p\,\mathrm{d}x = 0.$$

For instance, let us take $U_1(t) = 0$, $U_2(t) = bt^2$. Then we have for finite t

$$u = bxt^2/L, \quad P = P_o \exp[-\gamma bt^3/(3L)], \quad \rho = \rho_o \exp[-bt^3/(3L)],$$
$$p = \rho_o[L^2/6 - x^2/2][b^2t^4/L^2 + 2bt/L] \exp[-bt^3/(3L)].$$

Clearly this solution is valid if

$$M^2 = \rho U_2^2/(\gamma P) \ll 1,$$
$$\epsilon = \|p\|/P = |p(L,t) - p(0,t)|/P \ll 1.$$

First we can assume that $|(\gamma-1)\, bt^3/3L| < 1$. Then in cases of both blowing ($b < 0$, $\mathrm{d}P/\mathrm{d}t > 0$) and suction ($b > 0$, $\mathrm{d}P/\mathrm{d}t < 0$) we must demand $\rho_o |b|\, tL/P_o \ll 1$ to ensure $M^2 \ll 1$, $\epsilon \ll 1$. When we take $|(\gamma-1)\, bt^3/3L| \gg 1$, our solution is valid only if $b < 0$ because then $M \to 0$ when $t \to \infty$ at any values of b, ρ_o, P_o.

The examples given demonstrate some specific effects which may be observed in a bounded volume under non-steady boundary conditions which change the average pressure P in particular.

3. Subsonic flow of heat-conducting viscous gas

Our model should be generalized if one takes into account possible effects of viscosity and heat conductivity, and this extension of the model can be made by applying the same general concept. We can offer the following system of equations for the simulation of non-steady subsonic viscous flows, by neglecting p in the equation of state:

$$\partial \rho \boldsymbol{u}/\partial t + (\rho \boldsymbol{u} \cdot \nabla)\, \boldsymbol{u} + \boldsymbol{u} \nabla \cdot (\rho \boldsymbol{u}) + \nabla p = \rho \boldsymbol{f} + \boldsymbol{w} + \boldsymbol{D}, \tag{46}$$

$$\partial \rho/\partial t + \nabla \cdot (\rho \boldsymbol{u}) = \xi \rho, \tag{47}$$

$$\partial s/\partial t + \boldsymbol{u} \cdot \nabla s = (R/P)\, [\nabla(\lambda \nabla T) + Q] + q_m, \tag{48}$$

$$F(s, P, \rho) = 0, \tag{49}$$

where $T = P/R\rho$, $\boldsymbol{D}$, λ, Q, q_m are the temperature, viscous force, coefficient of molecular thermal conductivity, density of continuously distributed heat sources, and entropy source due to non-zero mass source, respectively. In (48) we have omitted the terms responsible for heat release due to viscous friction; these terms are usually too small in essentially subsonic flow. If we denote the dynamic viscosity as $\mu(P, T)$, then the components of $\boldsymbol{D}$ in Cartesian coordinates take the form

$$D_i = (\partial/\partial x_k)\, [\mu(\partial u_i/\partial x_k + \partial u_k/\partial x_i - \tfrac{2}{3} \delta_{ik}\, \partial u_j/\partial x_j)], \quad i, j, k = 1, 2, 3.$$

This expression can be simplified when $\nabla \mu = 0$ to

$$\boldsymbol{D} = \mu \Delta \boldsymbol{u} + \tfrac{1}{3} \mu \operatorname{grad} \operatorname{div} \boldsymbol{u}.$$

Some changes must also be introduced into the set of boundary conditions. Relations like (9) can be assigned, along with additional conditions which arise because of viscosity and heat conductivity. For example, on a permeable (porous) non-moving wall with suction ($u_n > 0$) or blowing ($u_n < 0$) we should use the following system of boundary conditions:

$$\left.\begin{array}{l} \Phi(u_n, P, \boldsymbol{r}, t) = 0, \quad u_{\tau 1} = 0, \quad u_{\tau 2} = 0, \\ \Psi(T, \boldsymbol{n} \nabla T, \boldsymbol{r}, t) = 0, \quad \boldsymbol{r} \in \Gamma, t \in J, \end{array}\right\} \tag{50}$$

where $u_{\tau 1}, u_{\tau 2}$ are the velocity components in two orthogonal tangential directions at the given boundary point, and Φ, Ψ are assigned functions.

Generally we could also use a similar set of boundary conditions at the inlet section of a duct with near-parallel flow. However, if we consider the more difficult case of imaginary boundary Γ_e which corresponds to the exit section (there we assign the pressure $P_e(t)$ averaged over Γ_e) through which the viscous vortical flow escapes from the confined duct (i.e. our domain G), we should use another set of relations. This is difficult, as in the inviscid case, but the application of our new model may simplify this problem, or at least its computational solution. In some cases we do not need to assign the distribution of p along Γ_e. Then we prescribe the pressure $P(t) = P_e(t)$, since $G = G_p$ in this case, and additionally we have to use some differential relations which could take the simplest form

$$\partial u_n/\partial n = 0, \quad \partial u_{\tau 1}/\partial n = 0, \quad \partial u_{\tau 2}/\partial n = 0, \quad \partial T/\partial n = 0$$

to complete the total number of necessary boundary conditions at Γ_e in the viscous case. If we must specify $\partial p/\partial n$ on Γ_e for a certain elliptic sub-problem within a general computational algorithm, it is possible to use equation (46) for this, or, in a primitive manner, even take $\partial p/\partial n = 0$ on Γ_e.

In order to deduce the new expression for $\operatorname{div} \boldsymbol{u}$, we denote the total entropy source as

$$q = (R/P)[\boldsymbol{\nabla}\cdot(\lambda \boldsymbol{\nabla} T) + Q] + q_m = -\boldsymbol{\nabla}[(\lambda/\rho^2)\boldsymbol{\nabla}\rho] + QR/P + q_m,$$

and then we arrive at the following equation that represents a logical extension of (22):

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \xi + \eta_m + (\gamma-1)[\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla} T) + Q]/(\gamma P) - (\mathrm{d}P/\mathrm{d}t)/(\gamma P), \tag{51}$$

where $\eta_m = q_m/c_p$.

If we use equation (51) instead of (48), then we obtain the following system, which is strictly equivalent to system (46)–(49) but now it excludes the entropy:

$$\partial\rho\boldsymbol{u}/\partial t + (\rho\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{u}\boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) + \boldsymbol{\nabla} p = \rho\boldsymbol{f} + \boldsymbol{w} + \boldsymbol{D}, \tag{52}$$

$$\partial\rho/\partial t + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{u}) = \xi\rho, \tag{53}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = \xi + \eta_m + (\gamma-1)[\boldsymbol{\nabla}\cdot(\lambda\boldsymbol{\nabla} T) + Q]/(\gamma P) - (\mathrm{d}P/\mathrm{d}t)/(\gamma P), \tag{54}$$

$$P = \rho R T, \quad \lambda = \lambda(P, T), \quad \mu = \mu(P, T). \tag{55}$$

Further we integrate (51) over the confined flow domain G with changing volume $V(t)$ by again applying Gauss' theorem in order to derive the following equation for the determination of $P(t)$:

$$V\,\mathrm{d}P/\mathrm{d}t = (\gamma-1)\int_\Gamma (\lambda\boldsymbol{n}\cdot\boldsymbol{\nabla} T)\,\mathrm{d}\sigma - \gamma P\int_\Gamma u_n\,\mathrm{d}\sigma + (\gamma-1)\int_G Q\,\mathrm{d}v + \gamma P\int_G (\xi+\eta_m)\,\mathrm{d}v. \tag{56}$$

This should be solved with the use of the independently assigned set of both initial and boundary conditions. Bearing in mind the inviscid version of the model, we can ignore equation (56) only if the special case $G = G_p$ is under consideration.

Thus we have obtained a closed system of equations (52)–(56) (of integro-differential type in the general case) for the set of flow variables $\{\boldsymbol{u}, p, \rho, T, P\}$ in $G \times J$. It is a radically new non-local model, including the Navier–Stokes equations, and is proposed for the simulation of unsteady subsonic flows in a bounded spatial domain where continuously distributed sources can be present along with appreciable heat fluxes.

One can confirm that this model is characterized by the complete exclusion of any acoustic effects, as in the case of its inviscid version. The local analysis of system (52)–(56) also excludes characteristics responsible for the propagation of sound disturbances because of the new form of the equation of state (49). Indeed, now we have an incompletely parabolic system where the second spatial derivatives, arising in the right-hand sides of (29)–(31), cannot lead to the existence of sound waves.

If $Q = \xi = \eta_m = 0$ and Γ is a non-moving impermeable wall, the average pressure

$P(t)$ depends only on the total heat flux through the whole boundary. This flux could be found either directly from boundary conditions (50) or in an implicit manner from the solution of the initial-boundary-value problem at the given time. The introduction of the global variable $P(t)$ results in the appearance of a new *degree of freedom* while considering the total mass balance in a closed volume with heat conduction through the walls. This model has notable advantages (first, we now have a better form of the continuity equation ensuring mass conservation) over the traditional models of incompressible fluid flow which are commonly used in various problems of heat convection.

The important distinctive feature of this model should be emphasized: $\operatorname{div} \boldsymbol{u} \neq 0$ even if $Q = \eta_m = \xi = \mathrm{d}P/\mathrm{d}t = 0$. This reflects the conformity of our model to the natural fact that any changes in the temperature field due to heat conduction must be accompanied by flow. For instance, it can induce a streaming, even if this is rather slow, caused by a non-equilibrium temperature field imposed on initially static fluid under steady pressure P. It should be recalled that within this model the smoothing of the pressure field by sound waves is supposed to occur much faster than the above changes in the fields of both temperature and density.

4. Exact solutions

Now we shall obtain a number of exact solutions, that simulate unsteady subsonic flows of viscous gas within our model. Some of these solutions reduce naturally to their inviscid versions. Below we shall not assign the boundary conditions in an independent manner, as should be done while posing some initial-boundary-value problems, but we shall try to find an exact solution of a system of equations immediately. Then the boundary values of all variables will automatically conform to this solution.

4.1. *Unsteady two-dimensional viscous near-adiabatic flows in a rectangular domain*

Let us consider two-dimensional viscous gas flow in the rectangular domain $G = \{|x| < L, |y| < H\}$, which corresponds to a confined flat duct with either permeable or impermeable sidewalls. Here we take $\boldsymbol{f} = 0$, $Q = 0$, $s = \text{const}$. The effects of molecular heat conduction are assumed to be negligible (this approximates the case of flow at substantial Reynolds numbers in a duct with thermally isolated walls). Then, introducing the vorticity ζ, we obtain the following system of governing equations:

$$\partial\zeta/\partial t + u\,\partial\zeta/\partial x + v\,\partial\zeta/\partial y + \zeta f = \nu\,\Delta\zeta, \tag{57}$$

$$\partial u/\partial x + \partial v/\partial y = -(\mathrm{d}P/\mathrm{d}t)/(\gamma P) = f, \tag{58}$$

$$\zeta = \partial v/\partial x - \partial u/\partial y, \quad \nu = \mu/\rho, \tag{59}$$

$$\rho = \rho(t) = \rho_o(P/P_o)^{1/\gamma}, \quad T = T(t) = T_o(P/P_o)^{(\gamma-1)/\gamma}, \tag{60}$$

and accordingly $\mu = \mu(t)$, $\nu = \nu(t)$, $f = f(t)$. Here we denote $P_o = P(0)$, $T_o = T(0)$, $\rho_o = \rho(0)$.

We shall look for the solutions of (57)–(60) which have the form

$$u = fx\alpha, \quad v = f\beta, \quad \alpha = \alpha(y), \quad \beta = \beta(y). \tag{61}$$

Then we require

$$\nabla\cdot\boldsymbol{u} = f(\alpha + \mathrm{d}\beta/\mathrm{d}y) = f, \quad \alpha + \mathrm{d}\beta/\mathrm{d}y = 1, \quad \zeta = -fx(\mathrm{d}\alpha/\mathrm{d}y), \tag{62}$$

and hence we derive from (57)

$$(\mathrm{d}f/\mathrm{d}t)[\mathrm{d}^2\beta/\mathrm{d}y^2] + f^2[\beta(\mathrm{d}^3\beta/\mathrm{d}y^3) + (2 - \mathrm{d}\beta/\mathrm{d}y)(\mathrm{d}^2\beta/\mathrm{d}y^2)] - \nu f(\mathrm{d}^4\beta/\mathrm{d}y^4) = 0. \tag{63}$$

(i) For the flow in a duct with impermeable sidewalls ($u = v = 0$ at $|y| = H = \pi$) we take

$$\alpha = 1+\cos y, \quad \beta = -\sin y.$$

Equations (62), (63) are valid if we find a solution of the nonlinear differential equation

$$\mathrm{d}f/\mathrm{d}t + 2f^2 + \nu f = 0. \tag{64}$$

If $\mu = \text{const}$ is assumed, we have to solve the second-order equation for $m = \ln\rho$:

$$\mathrm{d}^2m/\mathrm{d}t^2 - 2(\mathrm{d}m/\mathrm{d}t)^2 + \mu(\mathrm{d}m/\mathrm{d}t)\exp(-m) = 0.$$

However, we can investigate the main features of this solution by assuming $\nu = \text{const}$ when the changes in $P(t)$ are relatively small. Then we obtain the following non-trivial solution of (64) with specified initial values $f_o = f(0)$:

$$f = \nu f_o/[(2f_o+\nu)\exp(\nu t) - 2f_o]. \tag{65}$$

The stability of this solution depends on the value of f_o. It is easy to verify that (65) is asymptotically stable for all $f_o > 0$. For $f_o < 0$ this solution will be stable only if $f_o > -\nu/2$. Here we can introduce the characteristic Reynolds number at $t = 0$, that is independent of L:

$$Re_o = 2\pi|f_o|/\nu.$$

Then we can rewrite the stability condition for $f_o < 0$ in another manner: this flow will be stable only if $Re_o < Re_* = \pi$. Thus this solution should be emphasized as a remarkable example which is unique in that the flow evolution can change radically in relation to the values of initial Reynolds number Re_o. Here one could recall the conclusion derived from (37), (38): generally $f < 0$ (or $\mathrm{d}P/\mathrm{d}t > 0$) may promote the flow instability.

(ii) Another solution from the family (61) where

$$\alpha = (2-\cosh y)/2, \quad \beta = (\sinh y)/2,$$

has been obtained for the flow in a plane duct with permeable walls $|y| = H = Ar\cosh 2 \approx 1.32$. Here we deduce the following differential equation for $f(t)$:

$$\mathrm{d}f/\mathrm{d}t + 2f^2 - \nu f = 0, \quad f_o = f(0).$$

A non-trivial solution of this equation can be written by changing the sign in front of ν in (65):

$$f = -\nu f_o/[(2f_o-\nu)\exp(-\nu t) - 2f_o].$$

On analysing this solution one can make sure that it is asymptotically stable only if $f_o > 0$, i.e. when there is a suction through the walls $|y| = H$.

(iii) Both the above solutions have corresponding versions for the case of inviscid flow ($\nu = 0$). However, taking the same α, β, we should use the expression

$$f = f_o/(1+2tf_o)$$

instead of (65). Such solutions will be stable only if $f_o > 0$.

(iv) The above solutions may be compared with the exact solution of system (57)–(60) at $P = \text{const}$, $\nabla\cdot\boldsymbol{u} = 0$, $\nu = \text{const}$, that has been found for the flow in a confined duct $\{|x| < L, |y| < H = \frac{1}{2}\pi\}$ with permeable walls $|y| = H$ (blowing if $f > 0$, and suction if $f < 0$). This solution has the form

$$u = fx\cos y, \quad v = -f\sin y, \quad f = f_o\exp(-\nu t).$$

It is clear that it is asymptotically stable at any f_o.

When assigning characteristic parameters ρ_o, P_o, L, f_o in all the above solutions one should demand that $M_o^2 = \rho_o(f_o L)^2/(\gamma P_o) \ll 1$ (usually then $|p/P| \ll 1$ as well) in order to ensure the validity of our model.

4.2. *One-dimensional unsteady flows of viscous heat-conducting gas in a domain of finite length*

A series of exact solutions will now be derived for one-dimensional unsteady subsonic viscous flow with strong heat conduction. In this case we take the version of (52)–(55) for $u(x,t)$, $\rho(x,t)$, $p(x,t)$, $P(t)$ keeping μ, λ, γ, R as constant coefficients and assuming $\boldsymbol{f} = 0$, $Q = 0$:

$$\rho(\partial u/\partial t + u\,\partial u/\partial x) + \partial p/\partial x = \tfrac{4}{3}\mu\,\partial^2 u/\partial x^2, \tag{66}$$

$$\partial \rho/\partial t + \partial(\rho u)/\partial x = 0, \tag{67}$$

$$\partial u/\partial x = [\lambda(\gamma-1)/(\gamma P)]\,\partial^2 T/\partial x^2 - (\mathrm{d}P/\mathrm{d}t)/(\gamma P), \tag{68}$$

$$P = \rho R T. \tag{69}$$

Here we can follow the same approach as in the one-dimensional inviscid case, i.e. first we find a certain exact solution for $u(x,t)$, $\rho(x,t)$, $P(t)$ by using only (67)–(69). Then it is possible to obtain $p(x,t)$ by integrating (66) with the use of (13).

(i) Let us look for the solutions where

$$T = Ax^2, \quad \rho = P/(RAx^2), \quad u = Bx, \quad A = A(t) > 0, \quad B = B(t), \tag{70}$$

within an interval $0 < x_1 < x < x_2$, $x_2 - x_1 = L$. Then equation (68) becomes

$$\partial u/\partial x = B(t) = 2\lambda(\gamma-1)\,A/(\gamma P) - (\mathrm{d}P/\mathrm{d}t)/(\gamma P). \tag{71}$$

Substitution of (70) into (67) yields

$$\mathrm{d}Y/\mathrm{d}t = YB \quad \text{where} \quad Y(t) = P/A. \tag{72}$$

From (71), (72) we can obtain the following equation which gives a relation between possible functions $P(t)$ and $A(t)$ that in turn enables us to obtain a family of exact solutions:

$$(\gamma+1)\,A\,\mathrm{d}Y/\mathrm{d}t + Y\,\mathrm{d}A/\mathrm{d}t - 2\lambda(\gamma-1)\,A = 0. \tag{73}$$

For instance, on specifying function $A(t)$ we get Y from the solution of (73). Then we can find $B(t)$ from (72) and so $u(x,t)$, $\rho(x,t)$, $p(x,t)$ will be determined. We see that for all such solutions

$$\partial^2 u/\partial x^2 = 0, \quad \partial u/\partial t + u\,\partial u/\partial x = x(B^2 + \mathrm{d}B/\mathrm{d}t) = (x/Y)\,\mathrm{d}^2 Y/\mathrm{d}t^2.$$

(ii) If we take $A = A_o \exp(\alpha t)$ with constant values of both A_o and $\alpha \neq 0$, equation (73) will have the form

$$(\gamma+1)\,\mathrm{d}Y/\mathrm{d}t + Y\alpha - 2\lambda(\gamma-1) = 0, \tag{74}$$

and the solution of (74) for finite t can be written as

$$Y = Y_o\,\omega + 2\lambda[(\gamma-1)/\alpha][1-\omega],$$

where $$\omega = \exp[-\alpha t/(\gamma+1)], \quad Y_o = P_o/A_o, \quad P_o = P(0).$$

As a result we have

$$P = P_o\,\omega\exp(\alpha t) + [2\lambda A_o(\gamma-1)/\alpha][1-\omega]\exp(\alpha t),$$

and then one can find $u(x,t)$, $\rho(x,t)$.

(iii) Let us consider the case $A = \text{const}$. Then we obtain directly from (73) that

$$\left.\begin{aligned} P &= P_o + t[2\lambda A(\gamma-1)/(\gamma+1)], \quad u = x(\mathrm{d}P/\mathrm{d}t)/P,\\ M^2 &= u^2/(\gamma RT) = 4A\lambda^2(\gamma-1)^2/[\gamma(\gamma+1)^2 RP^2]. \end{aligned}\right\} \tag{75}$$

It is easy to see that $\mathrm{d}P/\mathrm{d}t > 0$, $\partial\rho/\partial t > 0$ because of the negative difference between boundary values of mass flux

$$\delta\rho u = \rho u(x_2, t) - \rho u(x_1, t) = -2\lambda(\gamma-1)\,L/[R(\gamma+1)\,x_1\,x_2] < 0.$$

(iv) If we take $P = \text{const}$, the following solution can be found from (73):

$$u = 2kAx, \quad A = A_o/[1+2kA_o\,t], \quad A_o = A(0), \quad k = \lambda(\gamma-1)/(\gamma P) > 0. \tag{76}$$

Here $\partial\rho/\partial t > 0$ due to $\delta\rho u < 0$ as in (75).

In this and the previous solution we have $\partial u/\partial t + u\,\partial u/\partial x = 0$ and hence $p(x,t) = 0$. Therefore solutions (75), (76) satisfy even the more general system of equations describing one-dimensional motion of a compressible viscous medium.

4.3. *One-dimensional flows of viscous heat-conducting gas in a half-space*

Below, a set of exact solutions is given for unsteady one-dimensional subsonic flows of viscous heat-conducting gas in the half-space $x > 0$. Here we assume $P = P_\infty = \text{const}$, $T_\infty = T_o = \text{const}$ for all these flows. Then we derive the following system of equations from (66)–(69):

$$\partial T/\partial t + u\,\partial T/\partial x - T\,\partial u/\partial x = 0,$$
$$\partial u/\partial x = k(\partial^2 T/\partial x^2),$$

where $k = \lambda(\gamma-1)/(\gamma P) = \text{const} > 0$. Now one can write the relation between u and T as follows:

$$u = k(\partial T/\partial x) + W,$$

with a function $W(t)$ which depends on the boundary conditions at $x = 0$ or $x = \infty$. Then, by eliminating u from the system, we obtain the nonlinear parabolic equation

$$\partial T/\partial t + k(\partial T/\partial x)^2 + W(\partial T/\partial x) - kT(\partial^2 T/\partial x^2) = 0. \tag{77}$$

Thus the problem reduces to the solving of this equation with the appropriate $W(t)$. It should be noted, however, that this equation departs substantially from the conventional equation of heat conduction, and hence new unusual effects can be studied within our approach.

If we consider only those solutions where $u_\infty = 0$, then $W(t) = 0$ and so equation (77) can be reduced to

$$\partial T/\partial t + k(\partial T/\partial x)^2 - kT(\partial^2 T/\partial x^2) = 0, \tag{78}$$

which in turn can be rewritten with the use of transformation $T = \exp(\Psi)$ as

$$\partial\Psi/\partial t - k(\partial^2\Psi/\partial x^2)\exp(\Psi) = 0.$$

(i) The following exact solution of (77) has been found for finite t:

$$T = g\exp(-\eta x) + T_o, \quad u = k\eta g[1-\exp(-\eta x)],$$
$$g = [A\exp(bt)]/[1+(A/T_o)(1-\exp(bt))],$$
$$b = k\eta^2 T_o, \quad \eta = \text{const} > 0, \quad A = \text{const}, \quad |A| \ll T_o, \quad 0 < t \lesssim 1/b.$$

This simulates the flow initiated by assigned variations of temperature on the non-moving impermeable wall $x = 0$, and results in $u_\infty = W(t) = k\eta g$.

Here it is logical to demand

$$M^2 = \|u^2/(\gamma R T_o)\| = O[(\lambda^2\eta^2 A^2)/(RP^2 T_o)] \ll 1. \quad (79)$$

(ii) In addition it is worth giving the simple stationary solution of (77):

$$T = T_o[1-\exp(-\eta x)], \quad u = -k\eta T,$$
$$\eta = \text{const} > 0, \quad x > x_o \gtrsim 1/\eta,$$

where requirement (79) with $A = O(T_o)$ has to be met as well.

(iii) In the case $u_\infty = 0$ the following solution of (78) has been found:

$$T = A\exp(tk\eta^2 T_o)\exp(-\eta x) + T_o, \quad u = -k\eta(T-T_o),$$
$$\eta = \text{const} > 0, \quad A = \text{const}, \quad |A| \ll T_o, \quad 0 < t \lesssim 1/(k\eta^2 T_o),$$

with the condition (79).

(iv) We have also obtained the solution of (78) with harmonic oscillations of both u and T at boundary $x = 0$:

$$T = T_o + A\exp[\mathrm{i}(\beta x - \omega t)], \quad u = \mathrm{i}k\beta(T-T_o),$$
$$\beta = (1+\mathrm{i})/\delta, \quad \delta = (2kT_o/\omega)^{1/2}, \quad A = \text{const}.$$

Here $|A| \ll T_o$, $\omega^2\delta^2/(RT_o) \ll 1$ are the obvious conditions for the model validity.

5. Concluding remarks

A new mathematical model has been proposed for the simulation of non-steady subsonic flows in a bounded spatial domain under rather complex boundary conditions, with continuously distributed sources of mass, momentum and entropy as well as with heat conduction, etc., but excluding all acoustic phenomena. This model is an extension of the classical model of incompressible fluid flow, the latter having too many restrictions to be applicable for the solution of the topical problems discussed in the introduction.

It should be emphasized that this model is essentially non-local. Indeed, we have departed from the traditional way of designing a local system of differential equations governing the fluid motion in a small internal volume irrespective of the possible set of boundary conditions. Rather, a quite general model of the initial-boundary-value problem has been created for the simulation of unsteady subsonic flows without acoustics. Through this concept we have obtained novel systems of integro-differential equations providing ample opportunities for their applications.

The approach described involves the procedure of pressure splitting, where the total static pressure $p_c(\boldsymbol{r}, t)$ is represented as a sum of two unknown variables: the average pressure $P(t)$ and the normalized pressure $p(\boldsymbol{r}, t)$. It has been shown that the global variable $P(t)$ depends on the assigned set of initial and boundary conditions as well as on the distribution of volume sources. Thereby this procedure differs radically from the usual methods where one considers small disturbances $p'(\boldsymbol{r}, t)$ near the known mean pressure $p_o(\boldsymbol{r})$ – for instance, those which are applied in atmospheric dynamics and acoustics, and in the theory of hydrodynamic stability, etc.

In the case of inviscid subsonic flow we have derived a new approximate system of basic equations which exhibits a number of distinct features. First, in comparison with the classical model of incompressible fluid flow this model is characterized by $\operatorname{div}\boldsymbol{u} \neq 0$. Moreover, the proposed system of equations displays non-local instantaneous connections between $\operatorname{div}\boldsymbol{u}$, boundary conditions and distribution of volume sources all over the spatial domain G. This is not unexpected, since the speed of sound is infinitely large within this model. As a result this approach gives the solution of the long-standing problem of simulation of continuously distributed sources of mass, momentum and entropy that was usually considered only by applying the general model of compressible fluid flow. The possibility of assigning the boundary values of normal velocity in an independent way is the other important advantage of this model. It has also been shown that time changes in $P(t)$ could be the reason for the appearance of vorticity sources which in turn could have a radical influence on the stability of all the flow.

After taking into account the effects of viscosity and molecular heat conductivity we have developed a more general version of the model that displays new unusual properties. For instance, equation (54) shows that $\operatorname{div} \boldsymbol{u}$ depends on $\operatorname{div}(\lambda \boldsymbol{\nabla} T)$ even if $\xi = \eta_m = Q = \mathrm{d}P/\mathrm{d}t = 0$. This significant feature removes the principal contradictions arising when we try to use the equation $\operatorname{div} \boldsymbol{u} = 0$ in subsonic problems with strong heat conduction. In particular, this approach enables one to eliminate the spurious effects of mass imbalance which can occur if the classical model of incompressible viscous fluid flow is applied to the study of heat convection in a closed volume. Thereby a radically new interpretation of the Navier–Stokes equations is given for a heat-conducting gaseous medium where acoustics is fully excluded. This model seems to be the most effective means in the study of high-temperature subsonic flows with distributed heat sources since the exclusion of sound waves permits us to investigate the direct interactions between the disturbances of vorticity and entropy, so that the influence of thermal effects on the hydrodynamic stability of bounded vortical flows can be found.

The absence of acoustic effects within this model may be of great value in the numerical simulation of unsteady subsonic flows. If $M \ll 1$, we can make the following estimates which are valid for many types of finite-difference schemes:

$$\tau = O(h/u_m), \quad \tau_c = O[h/(a+u_m)] \ll \tau.$$

Here τ, τ_c are the time steps of the finite-difference schemes for the given model and for the general model of a compressible medium, respectively, h is the minimum step size of the spatial grid, $u_m = \max|\boldsymbol{u}|$ in G, a is the mean value of sound velocity. The same holds true for viscous flows if we neglect the influence of viscosity and heat conductivity on the time step, as is acceptable at appreciable Reynolds numbers.

In addition, our model can simplify some computational problems in assigning the appropriate set of boundary conditions, for both viscous and inviscid cases. Generally, by excluding acoustics, we have avoided the numerous difficulties which could arise while analysing the acoustic properties of permeable boundaries.

The set of exact solutions given demonstrates how this approach could be used for the simulation of diverse subsonic flows, both viscous and inviscid. All these solutions represent a necessary supplement to the general description of the model. Additionally, they can serve as tests if we decide to apply this model to the computational study of multi-dimensional unsteady problems. Moreover, every exact solution of a nonlinear problem in fluid mechanics, especially any time-dependent solution of Navier–Stokes equations, has a high intrinsic value irrespective of its practical interpretation.

It is difficult to enumerate the great number of cases where this model could be successfully applied, but it is apparent that internal subsonic problems (unsteady non-uniform flows in ducts or in closed volumes under complex boundary conditions, with both distributed sources and intensive heat transfer, etc.) represent a vast area of possible applications. This model, being much more general than the classical model of incompressible fluid flow, takes into account many delicate effects incompletely studied before. It can promote new insight into some fundamental phenomena, and in turn, effective means of control over bounded subsonic flows could be developed.

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